The matrix theory of automorphisms for twisted affine Kac-Moody algebras $A_\ell^{(2)}$ (for $\ell \geqslant 2$) and $D_\ell^{(2)}$ (for $\ell \geqslant 3$)

1998 J. Phys. A: Math. Gen. 31 9703

(http://iopscience.iop.org/0305-4470/31/48/007)

J. Phys. A: Math. Gen. **31** (1998) 9703–9714. Printed in the UK PII: S0305-4470(98)94179-8

The matrix theory of automorphisms for twisted affine Kac–Moody algebras $A_\ell^{(2)}$ (for $\ell \geqslant 2$) and $D_\ell^{(2)}$ (for $\ell \geqslant 3$)

S P Clarke and J F Cornwell†

School of Physics and Astronomy, University of St Andrews, North Haugh, St Andrews, Fife KY16 9SS, UK

Received 15 May 1998, in final form 21 September 1998

Abstract. The application to the two series of classical *twisted* affine Kac–Moody algebras $A_\ell^{(2)}$ (for $\ell \geqslant 2$) and $D_\ell^{(2)}$ (for $\ell \geqslant 3$) of the matrix formulation of automorphisms developed earlier for *untwisted* affine Kac–Moody algebras is investigated. The conjugacy classes of involutive automorphisms and the corresponding real forms are deduced explicitly for the special case of $A_2^{(2)}$.

1. Introduction

A comprehensive method of dealing with all the automorphisms of an *untwisted* affine Kac–Moody algebra based on a matrix formulation of the untwisted affine Kac–Moody algebras has been developed by Cornwell [1]. The aim of this paper is to demonstrate how this method applies to the two great series of classical *twisted* affine Kac–Moody algebras $A_\ell^{(2)}$ (for $\ell \geqslant 2$) and $D_\ell^{(2)}$ (for $\ell \geqslant 3$). The simplest case, $A_2^{(2)}$, will be treated in detail.

As is well known, the study of the involutive automorphisms of complex semi-simple Lie algebras by Gantmacher [2] allowed Gantmacher [3] to obtain a very elegant systematic determination of all the simple real Lie algebras. As the situation for the affine Kac–Moody algebras is similar, the determination the conjugacy classes of the group of automorphisms of these algebras will yield their real forms. This will be done explicitly here for $A_2^{(2)}$.

In addition, there are also significant implications for the Virasoro algebra and conformal field theory through the Sugawara construction. The part played by Kac–Moody automorphisms in this context has been described, for example, by Bernard [4], Walton [5], Bouwknegt [6], and Font [7].

In a set of previous papers, Cornwell [8–10] and Clarke and Cornwell [11, 12] have discussed in detail the conjugacy classes of the involutive automorphisms of the four series of untwisted algebras $A_\ell^{(1)}$, $B_\ell^{(1)}$, $C_\ell^{(1)}$, and $D_\ell^{(1)}$. (It is interesting that the recent Sakate diagram method of Pati and Parashar [13] produces results in complete agreement with these for the two cases of $\mathrm{A}_1^{(1)}$ and $\mathrm{A}_2^{(1)}$ that they have studied in detail.)

The complex affine Kac–Moody algebras have structures that are now very well established. For reviews see Kac [14], Goddard and Olive [15, 16], and Cornwell [17]. Unless otherwise stated all the notations and conventions that will be employed in this paper are those of the latter reference. In particular, quantities belonging to the simple complex Lie algebra $\tilde{\mathcal{L}}^0$ associated with an untwisted affine Kac–Moody algebra $\tilde{\mathcal{L}}^{(1)}$ are distinguished from the corresponding quantities belonging to $\tilde{\mathcal{L}}^{(1)}$ by a superscript 0, so

† E-mail address: jfc@st-and.ac.uk

that, for example, α is the linear functional on the Cartan subalgebra $\mathcal{H}$ of $\tilde{\mathcal{L}}^{(1)}$ that is the extension of the linear functional α^0 on the Cartan subalgebra $\mathcal{H}^0$ of $\tilde{\mathcal{L}}^0$.

The organization of this paper is as follows. In section 2 the essential facts concerning the construction of untwisted and twisted affine Kac–Moody algebras are briefly summarized, and it is shown that for the elements of $A_\ell^{(2)}$ (for $\ell \geqslant 2$) and $D_\ell^{(2)}$ (for $\ell \geqslant 3$) a special condition (14) always holds. The relevant features of the matrix formulation of automorphisms of untwisted affine Kac–Moody algebras are then succinctly presented in section 3, where it is shown that the condition (14) plays a vital role in the extension of this method to $A_\ell^{(2)}$ (for $\ell \geqslant 2$) and $D_\ell^{(2)}$ (for $\ell \geqslant 3$). Finally, in section 4, after presenting the explicit basis elements of $A_2^{(2)}$, the resulting conjugacy classes of involutive automorphisms of $A_2^{(2)}$ and the corresponding real forms of $A_2^{(2)}$ are derived.

2. The structure of affine Kac–Moody algebras

2.1. Untwisted affine Kac–Moody algebras

Every complex *untwisted* affine Kac–Moody algebra $\tilde{\mathcal{L}}^{(1)}$ can be constructed from its corresponding simple complex Lie algebra $\tilde{\mathcal{L}}^0$ (assumed to be of rank ℓ^0) by the following well known construction [14–17] in which $\tilde{\mathcal{L}}^{(1)}$ may be taken to have a general element of the form

$$\sum_{j=-\infty}^{\infty}\sum_{r=1}^{n^0}\mu_{jr}t^j\otimes a_r^0+\mu_c c+\mu_d d \qquad (1)$$

where μ_{jr}, μ_c, and μ_d are arbitrary complex numbers, with only a finite number of the μ_{jr} being non-zero. Here j takes any integer value, a_r^0 are the basis elements of $\tilde{\mathcal{L}}^0$ (where $r = 1, 2, \ldots, n^0$, n^0 being the order of $\tilde{\mathcal{L}}^0$), and where t is a complex number. The commutators of $\tilde{\mathcal{L}}^{(1)}$ are given by

$$[t^j\otimes a^0, t^k\otimes b^0]=t^{j+k}\otimes[a^0,b^0]+j\delta^{j+k,0}B^0(a^0,b^0)c \qquad (2)$$

(for all integers j and k and all $a^0, b^0 \in \tilde{\mathcal{L}}^0$, where the commutators and Killing form $B^0(\,,\,)$ of the right-hand side of (2) are those of $\tilde{\mathcal{L}}^0$),

$$[t^j\otimes a^0, c]=0 \qquad (3)$$

(for all integers j and all $a^0 \in \tilde{\mathcal{L}}^0$),

$$[d, t^j\otimes a^0]=jt^j\otimes a^0 \qquad (4)$$

(for all integers j and all $a^0 \in \tilde{\mathcal{L}}^0$), and

$$[d, c]=0. \qquad (5)$$

If $h^0_{\alpha_k^0}$ (for $k = 1, 2, \ldots, \ell^0$) are the Weyl basis elements of the Cartan subalgebra $\mathcal{H}^0$ of $\tilde{\mathcal{L}}^0$ corresponding to the simple roots α_k^0 (for $k = 1, 2, \ldots, \ell^0$) of $\tilde{\mathcal{L}}^0$, then the $\ell^0 + 2$ basis elements c, d, and $t^0 \otimes h^0_{\alpha_k^0}$ (for $k = 1, 2, \ldots, \ell^0$) constitute a basis for a Cartan subalgebra $\mathcal{H}$ of $\tilde{\mathcal{L}}^{(1)}$.

Every linear function, and, in particular, every root, α^0 that is defined on $\mathcal{H}^0$ can be 'extended' to give a linear functional α on $\mathcal{H}$ by the definitions

$$\alpha(t^0\otimes h^0_{\alpha_k^0})=\alpha^0(h^0_{\alpha_k^0}) \text{ (for } k=1,2,\ldots,\ell^0) \qquad \alpha(c)=0,\ \alpha(d)=0. \qquad (6)$$

Moreover, if δ is the linear functional on $\mathcal{H}$ defined by

$$\delta(t^0 \otimes h^0_{\alpha^0_k}) = 0 \text{ (for } k = 1, 2, \ldots, \ell^0) \qquad \delta(c) = 0,\ \delta(d) = 1 \tag{7}$$

then, for every integer value of j, $t^j \otimes e^0_{\alpha^0}$ corresponds to a root $j\delta + \alpha$ of $\tilde{\mathcal{L}}$ and $t^j \otimes h^0_{\alpha^0_k}$ corresponds to a root $j\delta$ of $\tilde{\mathcal{L}}$ (for $k = 1, 2, \ldots, \ell^0$). Moreover, for $k = 1, 2, \ldots, \ell^0$ the simple roots α_k of $\tilde{\mathcal{L}}^{(1)}$ are just the extensions of the simple roots α^0_k of $\tilde{\mathcal{L}}^0$ and the remaining simple root α_0 of $\tilde{\mathcal{L}}^{(1)}$ is given by $\alpha_0 = \delta - \alpha_H$, where α_H is the extension of the highest root α^0_H of $\tilde{\mathcal{L}}^0$.

The compact real form $\tilde{\mathcal{L}}^{(1)}_c$ of $\tilde{\mathcal{L}}^{(1)}$ can be taken to have the basis elements:

$$\begin{array}{llll} \mathrm{i}c, \mathrm{i}d, \mathrm{i}h_{\alpha_k} & (k = 1, 2, \ldots, \ell^0) & & \\ e_\alpha + e_{-\alpha} & \mathrm{i}(e_\alpha - e_{-\alpha}) & (\alpha \in \Delta^0_+) & \\ e^k_{j\delta} + e^k_{-j\delta} & \mathrm{i}(e^k_{j\delta} - e^k_{-j\delta}) & (k = 1, 2, \ldots, \ell^0;\ j = \pm 1, \pm 2, \ldots) & \\ e_{j\delta+\alpha} + e_{-(j\delta+\alpha)} & \mathrm{i}(e_{j\delta+\alpha} - e_{(j\delta+\alpha)}) & (\alpha \in \Delta^0;\ j = 1, 2, \ldots). & \end{array} \tag{8}$$

Here

$$h_{\alpha_k} = t^0 \otimes h^0_{\alpha^0_k} \qquad e_{j\delta+\alpha} = t^j \otimes e^0_{\alpha^0} \qquad e^k_{j\delta} = it^j \otimes h^0_{\alpha^0_k}. \tag{9}$$

Let $\boldsymbol{\Gamma}$ be a faithful irreducible representation of some dimension $d_{\boldsymbol{\Gamma}}$ of $\tilde{\mathcal{L}}^0$. Then the first term $\sum_{j=-\infty}^{\infty} \sum_{r=1}^{n^0} \mu_{jr} t^j \otimes a^0_r$ of the general element (1) of the affine untwisted Kac–Moody algebra $\tilde{\mathcal{L}}^{(1)}$ is represented by the $d_{\boldsymbol{\Gamma}} \times d_{\boldsymbol{\Gamma}}$ matrix $\sum_{j=-\infty}^{\infty} \sum_{r=1}^{n^0} \mu_{jr} t^j \boldsymbol{\Gamma}(a^0_r)$. A typical matrix of this form will be denoted by $\mathbf{a}(t)$, i.e.

$$\mathbf{a}(t) = \sum_{j=-\infty}^{\infty} \sum_{r=1}^{n^0} \mu_{jr} t^j \boldsymbol{\Gamma}(a^0_r). \tag{10}$$

Clearly all the entries of $\mathbf{a}(t)$ are Laurent polynomials in the complex variable t. A typical element of $\tilde{\mathcal{L}}$ can then be written as

$$\mathbf{a}(t) + \mu_c c + \mu_d d. \tag{11}$$

(Of course in *no* sense do the + signs in (11) represent ordinary matrix addition.)

2.2. Construction of the complex twisted affine Kac–Moody algebras

The following construction of *twisted* affine Kac–Moody algebras from untwisted affine Kac–Moody algebras is also well known [14–17]. Suppose that $\tilde{\mathcal{L}}^0$ is a simple complex Lie algebra that possesses an *outer* automorphism, and let $\tilde{\mathcal{L}}^{(1)}$ be the untwisted affine Kac–Moody algebra constructed from $\tilde{\mathcal{L}}^0$ by the procedure of the previous subsection. (The only possible choices of $\tilde{\mathcal{L}}^0$ are A_{ℓ^0} (for $\ell^0 \geqslant 2$), D_{ℓ^0} (for $\ell^0 \geqslant 3$), and E_6.) Let τ be a 'rotation' of the set of roots Δ^0 of $\tilde{\mathcal{L}}^0$ and let ψ_τ be the corresponding outer automorphism of $\tilde{\mathcal{L}}^0$. For all of these $\tilde{\mathcal{L}}^0$ (except $\tilde{\mathcal{L}}^0 = D_4$) there is only one non-trivial τ, and this satisfies $\tau(\tau(\alpha_k)) = \alpha_k$ for $k = 1, 2, \ldots, \ell^0$. For example, for A_{ℓ^0} (for $\ell^0 \geqslant 2$), $\tau(\alpha_k) = \alpha_{\ell^0+1-k}$ for $k = 1, 2, \ldots, \ell^0$. For $\tilde{\mathcal{L}}^0 = D_4$ there are a number of such two-fold rotations, but there are also rotations for which $\tau(\tau(\tau(\alpha_k))) = \alpha_k$ for $k = 1, 2, 3, 4$. The situation may be summarized by the statement that $\tau^q = 1$, where $q = 2$ or 3, the latter occurring only for the three-fold rotations of D_4. For the corresponding outer automorphisms of $\tilde{\mathcal{L}}^0$, $(\psi_\tau)^q = 1$, implying that ψ_τ has q different eigenvalues, namely $\mathrm{e}^{2\pi \mathrm{i} p/q}$ for $p = 0, \ldots, q-1$. The elements of $\tilde{\mathcal{L}}^0$ may be arranged as eigenvectors of ψ_τ. Let $\tilde{\mathcal{L}}^{0(q)}_p$ be the eigenspace corresponding to eigenvalue $e^{2\pi i p/q}$, and let n_p^0 be its dimension. Then $a \in \tilde{\mathcal{L}}_p^{0(q)}$ if $a \in \tilde{\mathcal{L}}^0$ and $\psi_\tau(a) = e^{2\pi i p/q} a$.

Consider the subalgebra $\tilde{\mathcal{L}}^{(q)}$ of $\tilde{\mathcal{L}}^{(1)}$ that consists of all complex linear combinations of c, d, and, for each $p = 0, \ldots, q-1$, of $t^j \otimes a_{pr}^0$ for all basis elements a_{pr}^0 of $\tilde{\mathcal{L}}_p^{0(q)}$ and all integers j such that $j \bmod q = p$. This may be succinctly expressed as

$$\tilde{\mathcal{L}}^{(q)} = (\mathbb{C}c) \oplus (\mathbb{C}d) \oplus \sum_{p=0}^{q-1} \sum_{j,\, j \bmod q = p} t^j \otimes \tilde{\mathcal{L}}_p^{0(q)}. \tag{12}$$

It is easily shown that $\tilde{\mathcal{L}}^{(q)}$ (for $q = 2$ or 3) is a twisted affine Kac–Moody algebra.

2.3. The classical twisted affine Kac–Moody algebras $A_\ell^{(2)}$ (for $\ell \geqslant 2$) and $D_\ell^{(2)}$ (for $\ell \geqslant 3$)

Now consider the situation in which the twisted affine Kac–Moody algebra is $A_\ell^{(2)}$ (for $\ell \geqslant 2$) or $D_\ell^{(2)}$ (for $\ell \geqslant 3$), which together give the complete set of classical examples of $\tilde{\mathcal{L}}^{(q)}$ with $q = 2$. In terms of the faithful irreducible representation $\boldsymbol{\Gamma}$ of dimension d_Γ of $\tilde{\mathcal{L}}^0$ introduced in the previous subsection, a general term of $\tilde{\mathcal{L}}^{(2)}$ of the form

$$\sum_{j=-\infty,\, j \text{ even}}^{\infty} \sum_{r=1}^{n_0^0} \mu_{j0r} t^j \otimes a_{0r}^0 + \sum_{j=-\infty,\, j \text{ odd}}^{\infty} \sum_{r=1}^{n_1^0} \mu_{j1r} t^j \otimes a_{1r}^0$$

is represented by the $d_\Gamma \times d_\Gamma$ matrix

$$\mathbf{a}(t) = \sum_{j=-\infty,\, j \text{ even}}^{\infty} \sum_{r=1}^{n_0^0} \mu_{j0r} t^j \boldsymbol{\Gamma}(a_{0r}^0) + \sum_{j=-\infty,\, j \text{ odd}}^{\infty} \sum_{r=1}^{n_1^0} \mu_{j1r} t^j \boldsymbol{\Gamma}(a_{1r}^0). \tag{13}$$

All the entries of $\mathbf{a}(t)$ are Laurent polynomials in the complex variable t. The analysis that follows depends on the crucial observation that there exists a $d_\Gamma \times d_\Gamma$ matrix $\mathbf{d}$ with the property that for *every* matrix $\mathbf{a}(t)$ of the form (13)

$$\mathbf{a}(t) = -\mathbf{d}\tilde{\mathbf{a}}(-t)\mathbf{d}^{-1}. \tag{14}$$

The proof that such a matrix $\mathbf{d}$ exists is as follows. As is well known, the Cartan involution ψ_C of the corresponding simple Lie algebra $\tilde{\mathcal{L}}^0$, defined by $\psi_C(h^0) = -h^0$ for all $h^0 \in \mathcal{H}^0$ and $\psi_C(e_{\alpha^0}^0) = e_{-\alpha^0}^0$ for every root α^0 of $\tilde{\mathcal{L}}^0$, is an outer automorphism of $\tilde{\mathcal{L}}^0$. However, the representation $\boldsymbol{\Gamma}$ of $\tilde{\mathcal{L}}^0$ can always be chosen so that $\boldsymbol{\Gamma}(h^0)$ is a real diagonal matrix (for all $h^0 \in \mathcal{H}^0$) and the $\boldsymbol{\Gamma}(e_{\alpha^0}^0)$ are real with $\boldsymbol{\Gamma}(e_{-\alpha^0}^0) = -\tilde{\boldsymbol{\Gamma}}(e_{\alpha^0}^0)$ (for every root α^0 of $\tilde{\mathcal{L}}^0$). With this choice, the action of ψ_C is given by

$$\psi_C(\boldsymbol{\Gamma}(a^0)) = -\tilde{\boldsymbol{\Gamma}}(a^0) \tag{15}$$

for all $a^0 \in \tilde{\mathcal{L}}^0$. As $(\psi_C)^2 = 1$ and $(\psi_\tau)^2 = 1$ (for $q = 2$), and as both ψ_C and ψ_τ are outer automorphisms of $\tilde{\mathcal{L}}^0$, it follows that ψ_τ must be the product of ψ_C with some inner automorphism ψ_i of $\tilde{\mathcal{L}}^0$. Suppose that this inner automorphism ψ_i is generated by a $d_\Gamma \times d_\Gamma$ unitary matrix $\mathbf{d}$, so that

$$\psi_i(\boldsymbol{\Gamma}(a^0)) = \mathbf{d}\boldsymbol{\Gamma}(a^0)\mathbf{d}^{-1} \tag{16}$$

for all $a^0 \in \tilde{\mathcal{L}}^0$. Combining (15) and (16) then gives

$$\psi_\tau(\boldsymbol{\Gamma}(a^0)) = -\mathbf{d}\tilde{\boldsymbol{\Gamma}}(a^0)\mathbf{d}^{-1} \tag{17}$$

for all $a^0 \in \tilde{\mathcal{L}}^0$, and hence

$$-\mathbf{d}\tilde{\boldsymbol{\Gamma}}(a^0_{kr})\mathbf{d}^{-1} = \begin{cases} \boldsymbol{\Gamma}(a^0_{kr}) & \text{for } k = 0 \\ -\boldsymbol{\Gamma}(a^0_{kr}) & \text{for } k = 1 \end{cases} \tag{18}$$

from which (14) follows immediately.

This argument also shows that a matrix $\mathbf{a}(t)$ of $\tilde{\mathcal{L}}^{(1)}$ of the form (10) is a member of $\tilde{\mathcal{L}}^{(2)}$ if and only if it satisfies the condition (14).

3. Matrix formulation of the automorphisms of affine Kac–Moody algebras

In the case of an *untwisted* affine Kac–Moody algebra $\tilde{\mathcal{L}}^{(1)}$ the matrix formulation of the automorphisms may be summarized in the following way [1]. Let $\boldsymbol{\Gamma}$ be a faithful irreducible representation of dimension $d_{\boldsymbol{\Gamma}}$ of $\tilde{\mathcal{L}}^0$, and let γ be the Dynkin index of $\boldsymbol{\Gamma}$. Then a typical element of $\tilde{\mathcal{L}}$ can then be written as in (11). Each of the four types (1a, 1b, 2a, and 2b) of automorphism of $\tilde{\mathcal{L}}^{(1)}$ depends on the following three quantities (although the dependence is different for the different types):

(1) a $d_{\boldsymbol{\Gamma}} \times d_{\boldsymbol{\Gamma}}$ matrix $\mathbf{U}(t)$, which is assumed to be invertible and for which all the entries of $\mathbf{U}(t)$ and $\mathbf{U}(t)^{-1}$ are assumed to be Laurent polynomials in the complex variable t;

(2) a non-zero complex parameter u;

(3) an arbitrary complex parameter ξ.

The precise actions of such automorphisms are as follows.

(1) Actions on $\mathbf{a}(t)$:

$$\phi(\mathbf{a}(t)) = \mathbf{U}(t)\Theta(\mathbf{a}(ut))\mathbf{U}(t)^{-1} + \frac{1}{\gamma}\mathrm{Res}\left\{\mathrm{tr}\left(\mathbf{U}(t)^{-1}\frac{\mathrm{d}\mathbf{U}(t)}{\mathrm{d}t}\Theta(\mathbf{a}(t))\right)\right\} c \tag{19}$$

where

$$\Theta(\mathbf{a}(t)) = \begin{cases} \mathbf{a}(ut) & \text{for type 1a} \\ -\tilde{\mathbf{a}}(ut) & \text{for type 1b} \\ \mathbf{a}(ut^{-1}) & \text{for type 2a} \\ -\tilde{\mathbf{a}}(ut^{-1}) & \text{for type 2b.} \end{cases} \tag{20}$$

(2) Actions on c and d:

$$\phi(c) = \mu c \tag{21}$$

and

$$\phi(d) = \mu\boldsymbol{\Phi}(\mathbf{U}(t)) + \xi c + \mu d \tag{22}$$

where $\boldsymbol{\Phi}(\mathbf{U}(t))$ is the $d_{\boldsymbol{\Gamma}} \times d_{\boldsymbol{\Gamma}}$ matrix that depends on $\mathbf{U}(t)$ according to the definition

$$\boldsymbol{\Phi}(\mathbf{U}(t)) = \left\{-t\frac{\mathrm{d}\mathbf{U}(t)}{\mathrm{d}t}\mathbf{U}(t)^{-1} + \frac{1}{d_{\boldsymbol{\Gamma}}}\,\mathrm{tr}\left(t\frac{\mathrm{d}\mathbf{U}(t)}{\mathrm{d}t}\mathbf{U}(t)^{-1}\right)\mathbf{1}\right\} \tag{23}$$

and

$$\mu = \begin{cases} 1 & \text{for type 1a and 1b} \\ -1 & \text{for type 2a and 2b.} \end{cases} \tag{24}$$

If the triples $\{\mathbf{U}(t), u, \xi\}$ and $\{\mathbf{U}'(t), u', \xi'\}$ specify two automorphisms of $\tilde{\mathcal{L}}^{(1)}$ of the *same* type, then these automorphisms of $\tilde{\mathcal{L}}^{(1)}$ are *identical* if and only if

$$u' = u \qquad \xi' = \xi \tag{25}$$

and there exists a non-zero complex number η and an integer k such that

$$\mathbf{U}'(t) = \eta t^k \mathbf{U}(t). \tag{26}$$

It is believed that the matrix formulation covers all the automorphisms, but this remains to be proved. Certainly the matrix formulation includes as special cases all the known ‘abstract’ examples of automorphisms, such as the Cartan involution and those associated with Dynkin diagram rotations and Weyl reflections [8–10]. Moreover, related investigations using other methods, such as the work of Pati and Parashar [13] mentioned in the introduction, have provided results that are in complete accord with those of the matrix formulation. In practical terms, the method is best suited to the affine Kac–Moody algebras based on the *classical* simple Lie algebras, as the faithful irreducible representations of the *exceptional* simple Lie algebras are so difficult to employ, not least because of the high dimensions.

Turning now to the situation for the *twisted* affine Kac–Moody algebras $\tilde{\mathcal{L}}^{(2)}$, where $\tilde{\mathcal{L}}^{(2)}$ is $A_\ell^{(2)}$(for $\ell \geqslant 2$) or $D_\ell^{(2)}$ (for $\ell \geqslant 3$), the first point to note is that $\tilde{\mathcal{L}}^{(2)}$ is a *subalgebra* of $\tilde{\mathcal{L}}^{(1)}$. Correspondingly, the matrices $\mathbf{a}(t)$ of (10) must be restricted to have the form (13), which, as noted previously, satisfy the constraint (14). This has the following very important consequences.

(1) An automorphism of $\tilde{\mathcal{L}}^{(1)}$ as described above acts as an automorphism of $\tilde{\mathcal{L}}^{(2)}$ for matrices $\mathbf{a}(t)$ that satisfy (10) if and only if

$$\mathbf{b}(t) = -\mathbf{d}\tilde{\mathbf{b}}(-t)\mathbf{d}^{-1} \tag{27}$$

where

$$\mathbf{b}(t) = \mathbf{U}(t)\Theta(\mathbf{a}(t))\mathbf{U}(t)^{-1}. \tag{28}$$

(2) It follows from (14) and (19) that for each of the algebras $A_\ell^{(2)}$ (for $\ell \geqslant 2$) and $D_\ell^{(2)}$ (for $\ell \geqslant 3$) the type 1b automorphism ψ associated with the triple $\{\mathbf{d}, -1, 0\}$ also acts as the *identity* automorphism.

(3) As noted in section 4.2 of Cornwell [1], if ϕ_1 and ϕ_2 are type 1b and type 1a automorphisms corresponding to the triples $\{\mathbf{U}_1(t), u_1, \xi_1\}$ and $\{\mathbf{U}_2(t), u_2, \xi_2\}$ respectively, then $\phi_1 \circ \phi_2$ is a type 1b automorphism corresponding to the triple $\{\mathbf{U}(t), u, \xi\}$, where

$$\mathbf{U}(t) = \mathbf{U}_1(t)\tilde{\mathbf{U}}_2(u_1 t)^{-1} \tag{29}$$

$$u = u_1 u_2 \tag{30}$$

and

$$\xi = \xi_1 + \xi_2 - \frac{1}{\gamma}\mathrm{Res}\left\{\mathrm{tr}\left(\mathbf{U}_1(t)^{-1}\frac{\mathrm{d}\mathbf{U}_1(t)}{\mathrm{d}t}\tilde{\mathbf{\Phi}}(\mathbf{U}_2(u_1 t))\right)\right\} \tag{31}$$

and where $\mathbf{\Phi}(\mathbf{U}(t))$ is defined in (23). Consequently, when $\tilde{\mathcal{L}}^{(2)}$ is $A_\ell^{(2)}$ (for $\ell \geqslant 2$) or $D_\ell^{(2)}$ (for $\ell \geqslant 3$), on taking ϕ_1 to be the identity automorphism ψ of the previous note, it follows that *every type 1b automorphism of* $\tilde{\mathcal{L}}^{(2)}$ *is identical to some type 1a automorphism of* $\tilde{\mathcal{L}}^{(2)}$ *and vice versa*.

(4) Similarly [1], if ϕ_1 and ϕ_2 are type 1b and type 2a automorphisms corresponding to the triples $\{\mathbf{U}_1(t), u_1, \xi_1\}$ and $\{\mathbf{U}_2(t), u_2, \xi_2\}$ respectively, then $\phi_1 \circ \phi_2$ is a type 2b automorphism corresponding to the triple $\{\mathbf{U}(t), u, \xi\}$, where $\mathbf{U}(t)$ and ξ are again given by (29) and (31), but where now $u = u_1^{-1}u_2$. Consequently, on again taking ϕ_1 to be the identity automorphism ψ, it follows that *every type 2b automorphism of* $\tilde{\mathcal{L}}^{(2)}$ *is identical to some type 2a automorphism of* $\tilde{\mathcal{L}}^{(2)}$ *and vice versa*. (Here $\tilde{\mathcal{L}}^{(2)}$ is again $A_\ell^{(2)}$ (for $\ell \geqslant 2$) or $D_\ell^{(2)}$ (for $\ell \geqslant 3$).)

The last two observations imply that for $A_\ell^{(2)}$ (for $\ell \geqslant 2$) and $D_\ell^{(2)}$ (for $\ell \geqslant 3$) it is unnecessary to consider the type 1b and type 2b automorphisms any further, and that it is sufficient to confine attention to the type 1a and type 2a automorphisms. For these, the condition (28), taken with Schur's lemma and the requirement that the entries in $\mathbf{U}(t)$ are all Laurent polynomials, implies that

$$\mathbf{d}\tilde{\mathbf{U}}(-t)\mathbf{d}^{-1}\mathbf{U}(t) = \lambda t^{\mu}\mathbf{1} \tag{32}$$

where $\lambda \in \mathbb{C}$ and μ is an integer. When taken with a requirement that the $\mathbf{U}(t)$ matrices correspond to Cartan preserving automorphisms, this imposes a severe constraint on the possible matrices $\mathbf{U}(t)$. Clearly both $\mathbf{1}$ and $\mathbf{d}$ provide possible $\mathbf{U}(t)$ matrices that satisfy (32), and is interesting that in the case of $A_2^{(2)}$ (which is considered in detail in section 4) these two solutions actually provide the representatives of *every* conjugacy class of involutive automorphism.

A type 1a automorphism corresponding to the triple $\{\mathbf{U}(t), u, \xi\}$ is *involutive* if and only if the following three conditions are all satisfied:

$$\mathbf{U}(t)\mathbf{U}(ut) = \eta t^{k}\mathbf{1} \tag{33}$$

for some complex number η and some integer k,

$$u^2 = 1 \tag{34}$$

and

$$\xi = -\frac{1}{2\gamma}\mathrm{Res}\left\{\mathrm{tr}\left(\mathbf{U}(t)^{-1}\frac{\mathrm{d}\mathbf{U}(t)}{\mathrm{d}t}\Phi(\mathbf{U}(ut))\right)\right\}. \tag{35}$$

Similarly a type 2a automorphism corresponding to the triple $\{\mathbf{U}(t), u, \xi\}$ is involutive if and only if the following two conditions are all satisfied:

$$\mathbf{U}(t)\mathbf{U}(ut^{-1}) = \eta t^{k}\mathbf{1} \tag{36}$$

for some complex number η and some integer k, and

$$\xi = -\frac{1}{2\gamma}\mathrm{Res}\left\{\mathrm{tr}\left(\mathbf{U}(t)^{-1}\frac{\mathrm{d}\mathbf{U}(t)}{\mathrm{d}t}\Phi(\mathbf{U}(ut^{-1}))\right)\right\}. \tag{37}$$

The necessary and sufficient conditions for the conjugacy of a pair of automorphisms ϕ_1 and ϕ_2 of $\tilde{\mathcal{L}}$ corresponding to the triples $\{\mathbf{U}_1(t), u_1, \xi_1\}$ and $\{\mathbf{U}_2(t), u_2, \xi_2\}$ respectively have been given by Cornwell [1]. Although they are essential in the detailed analysis, they are not necessary for understanding the results that emerge and which are given in subsequent sections, and so they will be omitted here. They do imply that in searching for the conjugacy classes of involutive automorphisms of $A_\ell^{(2)}$ (for $\ell \geqslant 2$) and $D_\ell^{(2)}$ (for $\ell \geqslant 3$) only three categories of class representatives need be considered, namely:

(1) type 1a involutive automorphisms with $u = 1$,

(2) type 1a involutive automorphisms with $u = -1$,

(3) type 2a involutive automorphisms with $u = 1$.

These considerations will be illustrated in the next section by examining in detail the the affine Kac–Moody algebra $A_2^{(2)}$.

4. Real forms and conjugacy classes of involutive automorphisms of the affine Kac–Moody algebra $A_2^{(2)}$

4.1. The twisted affine Kac–Moody algebra $A_2^{(2)}$

For $\tilde{\mathcal{L}}^0 = A_2$, $\tau(\alpha_1^0) = \alpha_2^0$ and $\tau(\alpha_2^0) = \alpha_1^0$, and

$$\begin{aligned}
&\psi_\tau(h^0_{\alpha_1^0}) = h^0_{\alpha_2^0} \qquad \psi_\tau(h^0_{\alpha_2^0}) = h^0_{\alpha_1^0} \\
&\psi_\tau(e^0_{\alpha_1^0}) = e^0_{\alpha_2^0} \qquad \psi_\tau(e^0_{\alpha_2^0}) = e^0_{\alpha_1^0} \qquad \psi_\tau(e^0_{\alpha_1^0+\alpha_2^0}) = -e^0_{\alpha_1^0+\alpha_2^0} \\
&\psi_\tau(e^0_{-\alpha_1^0}) = e^0_{-\alpha_2^0} \qquad \psi_\tau(e^0_{-\alpha_2^0}) = e^0_{-\alpha_1^0} \qquad \psi_\tau(e^0_{-(\alpha_1^0+\alpha_2^0)}) = -e^0_{-(\alpha_1^0+\alpha_2^0)}.
\end{aligned} \tag{38}$$

(See, for example, Cornwell [17] for details.) Consequently $\tilde{\mathcal{L}}_0^{0(2)}$ has basis elements

$$h^0_{\alpha_1^0} + h^0_{\alpha_2^0} \qquad e^0_{\alpha_1^0} + e^0_{\alpha_2^0} \qquad e^0_{-\alpha_1^0} + e^0_{-\alpha_2^0} \tag{39}$$

whereas the subspace $\tilde{\mathcal{L}}_1^{0(2)}$ has basis elements

$$h^0_{\alpha_1^0} - h^0_{\alpha_2^0} \qquad e^0_{\alpha_1^0} - e^0_{\alpha_2^0} \qquad e^0_{-\alpha_1^0} - e^0_{-\alpha_2^0} \qquad e^0_{\alpha_1^0+\alpha_2^0} \qquad e^0_{-(\alpha_1^0+\alpha_2^0)}. \tag{40}$$

Thus, by (12), $A_2^{(2)}$ has basis elements:

$$\begin{aligned}
&c, d \\
&t^j \otimes (h^0_{\alpha_1^0} + h^0_{\alpha_2^0}) \qquad t^j \otimes (e^0_{\alpha_1^0} + e^0_{\alpha_2^0}) \qquad t^j \otimes (e^0_{-\alpha_1^0} + e^0_{-\alpha_2^0}) \qquad \text{for } j \text{ even} \\
&t^j \otimes (h^0_{\alpha_1^0} - h^0_{\alpha_2^0}) \qquad t^j \otimes (e^0_{\alpha_1^0} - e^0_{\alpha_2^0}) \qquad t^j \otimes (e^0_{-\alpha_1^0} - e^0_{-\alpha_2^0}) \qquad \text{for } j \text{ odd} \\
&t^j \otimes e^0_{\alpha_1^0+\alpha_2^0} \qquad t^j \otimes e^0_{-(\alpha_1^0+\alpha_2^0)} \qquad \text{for } j \text{ odd.}
\end{aligned} \tag{41}$$

That is, by (9), $A_2^{(2)}$ has basis elements:

$$\begin{aligned}
&c,\ d,\ h_{\alpha_1} + h_{\alpha_2} \\
&(e^1_{j\delta} + e^2_{j\delta}) \qquad \text{for } j = \pm 2, \pm 4, \ldots \\
&(e_{j\delta+\alpha_1} + e_{j\delta+\alpha_2}) \qquad (e_{j\delta-\alpha_1} + e_{j\delta-\alpha_2}) \qquad \text{for } j = 0, \pm 2, \pm 4, \ldots \\
&(e^1_{j\delta} - e^2_{j\delta}) \qquad (e_{j\delta+\alpha_1} - e_{j\delta+\alpha_2}) \qquad (e_{j\delta-\alpha_1} - e_{j\delta-\alpha_2}) \qquad \text{for } j = \pm 1, \pm 3, \ldots \\
&e_{j\delta+\alpha_1+\alpha_2} \qquad e_{j\delta-\alpha_1-\alpha_2} \qquad \text{for } j = \pm 1, \pm 3, \ldots.
\end{aligned} \tag{42}$$

Let $\mathbf{\Gamma}$ be the three-dimensional irreducible representation of A_2 in which

$$\begin{aligned}
&\mathbf{\Gamma}(h^0_{\alpha_1^0}) = \mathbf{h}^0_{\alpha_1^0} = \tfrac{1}{6}(\mathbf{e}_{11} - \mathbf{e}_{22}) \qquad \mathbf{\Gamma}(h^0_{\alpha_2^0}) = \mathbf{h}^0_{\alpha_2^0} = \tfrac{1}{6}(\mathbf{e}_{22} - \mathbf{e}_{33}) \\
&\mathbf{\Gamma}(e^0_{\alpha_1^0}) = \mathbf{e}^0_{\alpha_1^0} = (\tfrac{1}{6})^{\frac{1}{2}}\mathbf{e}_{12} \qquad \mathbf{\Gamma}(e^0_{-\alpha_1^0}) = \mathbf{e}^0_{-\alpha_1^0} = -(\tfrac{1}{6})^{\frac{1}{2}}\mathbf{e}_{21} \\
&\mathbf{\Gamma}(e^0_{\alpha_2^0}) = \mathbf{e}^0_{\alpha_2^0} = (\tfrac{1}{6})^{\frac{1}{2}}\mathbf{e}_{23} \qquad \mathbf{\Gamma}(e^0_{-\alpha_2^0}) = \mathbf{e}^0_{-\alpha_2^0} = -(\tfrac{1}{6})^{\frac{1}{2}}\mathbf{e}_{32} \\
&\mathbf{\Gamma}(e^0_{\alpha_1^0+\alpha_2^0}) = \mathbf{e}^0_{\alpha_1^0+\alpha_2^0} = (\tfrac{1}{6})^{\frac{1}{2}}\mathbf{e}_{13} \qquad \mathbf{\Gamma}(e^0_{-(\alpha_1^0+\alpha_2^0)}) = \mathbf{e}^0_{-(\alpha_1^0+\alpha_2^0)} = -(\tfrac{1}{6})^{\frac{1}{2}}\mathbf{e}_{31}
\end{aligned} \tag{43}$$

where $\mathbf{e}_{jk}$ are the 3×3 matrices defined by $(\mathbf{e}_{jk})_{rs} = \delta_{jr}\delta_{ks}$ for $j, k, r, s = 1, 2, 3$. The value of the Dynkin index of this representation is given by $\gamma = \frac{1}{6}$. The explicit form of the matrix $\mathbf{d}$ with the crucial property (14) is

$$\mathbf{d} = \mathbf{d}^{-1} = \mathbf{e}_{13} - \mathbf{e}_{22} + \mathbf{e}_{31}. \tag{44}$$

The compact real form $\tilde{\mathcal{L}}_c^{(2)}$ of $\tilde{\mathcal{L}}^{(2)} = A_2^{(2)}$ may be taken to be the intersection of $\tilde{\mathcal{L}}_c^{(1)}$ with $\tilde{\mathcal{L}}^{(2)}$, so its basis elements may be taken to be:

$$
\begin{array}{ll}
\mathrm{i}c,\ \mathrm{i}d,\ \mathrm{i}(h_{\alpha_1}+h_{\alpha_2}),\ (e_{\alpha_1}+e_{\alpha_2}+e_{-\alpha_1}+e_{-\alpha_2}),\ \mathrm{i}(e_{\alpha_1}+e_{\alpha_2}-e_{-\alpha_1}-e_{-\alpha_2}) & \\
(e^1_{j\delta}+e^2_{j\delta}+e^1_{-j\delta}+e^2_{-j\delta}),\ \mathrm{i}(e^1_{j\delta}+e^2_{j\delta}-e^1_{-j\delta}-e^2_{-j\delta}) \qquad \text{for } j=2,4,\ldots & \\
\left.\begin{array}{l}(e_{j\delta+\alpha_1}+e_{j\delta+\alpha_2}+e_{-j\delta-\alpha_1}+e_{-j\delta-\alpha_2})\\ \mathrm{i}(e_{j\delta+\alpha_1}+e_{j\delta+\alpha_2}-e_{-j\delta-\alpha_1}-e_{-j\delta-\alpha_2})\end{array}\right\} \quad \text{for } j=\pm2,\ \pm4,\ldots & \\
(e^1_{j\delta}-e^2_{j\delta}+e^1_{-j\delta}-e^2_{-j\delta}),\ \mathrm{i}(e^1_{j\delta}-e^2_{j\delta}-e^1_{-j\delta}+e^2_{-j\delta}) \qquad \text{for } j=1,3,\ldots & \\
\left.\begin{array}{l}(e_{j\delta+\alpha_1}-e_{j\delta+\alpha_2}+e_{-j\delta-\alpha_1}-e_{-j\delta-\alpha_2})\\ \mathrm{i}(e_{j\delta+\alpha_1}-e_{j\delta+\alpha_2}-e_{-j\delta-\alpha_1}+e_{-j\delta-\alpha_2})\\ (e_{j\delta+\alpha_1+\alpha_2}+e_{-j\delta-\alpha_1-\alpha_2}),\ \mathrm{i}(e_{j\delta+\alpha_1+\alpha_2}-e_{-j\delta-\alpha_1-\alpha_2})\end{array}\right\} \quad \text{for } j=\pm1,\ \pm3,\ldots. &
\end{array}
\tag{45}
$$

Suppose that ψ is an involutive automorphism of $\tilde{\mathcal{L}}^{(2)}$, and hence of $\tilde{\mathcal{L}}_c^{(2)}$, so that the only eigenvalues of ψ are ±1. The basis elements of the corresponding real form $\mathcal{L}$ of $\tilde{\mathcal{L}}^{(2)}$ can then be constructed as follows: If $a \in \tilde{\mathcal{L}}_c^{(2)}$ is an eigenvector of ψ with eigenvalue $+1$, then $a \in \mathcal{L}$, whereas if $a \in \tilde{\mathcal{L}}_c^{(2)}$ is an eigenvector of ψ with eigenvalue -1, then $\mathrm{i}a \in \mathcal{L}$.

4.2. Conjugacy classes of involutive automorphisms and real forms of $A_2^{(2)}$

There are essentially two stages in determining the conjugacy classes of $A_2^{(2)}$ in the matrix formulation. The first is to find, for each type of automorphism, all the U(t) matrices corresponding to Cartan preserving involutive automorphisms. This involves working through all the root preserving transformations of $\tilde{\mathcal{L}}^0$ in the manner described in detail previously [8–11]. The second stage, which is rather more difficult, is to determine which of the corresponding automorphisms of $\tilde{\mathcal{L}}$ are actually conjugate to each other, and which are not. The long and detailed analysis that is involved will be omitted here, and so only the results will be described. The basis elements of each real form are displayed as eigenvectors of the corresponding involutive automorphism, thereby describing compactly both the involutive automorphism and its real form in a single set of statements.

1. Type 1a involutive automorphisms with $u = 1$ and associated real forms of $A_2^{(2)}$

(i) The conjugacy class with representative given by $\mathbf{U}(t) = \mathbf{1}_3$ and $\xi = 0$ consists only of the identity automorphism, for which the corresponding real form is the compact real form of (45).

(ii) There exists a class of involutive automorphisms with representative given by $\mathbf{U}(t) = \mathbf{d}$ and $\xi = 0$, where $\mathbf{d}$ is given by (44). For the associated real form the eigenvectors of the corresponding involutive automorphism ψ are:

(a) with eigenvalue $+1$:

1. $\mathrm{i}c$, $\mathrm{i}d$, $(e_{\alpha_1}+e_{\alpha_2}+e_{-\alpha_1}+e_{-\alpha_2})$;
2. $(e_{j\delta+\alpha_1}+e_{j\delta+\alpha_2}+e_{-j\delta-\alpha_1}+e_{-j\delta-\alpha_2}+e_{j\delta-\alpha_1}+e_{j\delta-\alpha_2}+e_{-j\delta+\alpha_1}+e_{-j\delta+\alpha_2})$ and $\mathrm{i}(e_{j\delta+\alpha_1}+e_{j\delta+\alpha_2}-e_{-j\delta-\alpha_1}-e_{-j\delta-\alpha_2}+e_{j\delta-\alpha_1}+e_{j\delta-\alpha_2}-e_{-j\delta+\alpha_1}-e_{-j\delta+\alpha_2})$ for $j = 2, 4, \ldots$;
3. $(e^1_{j\delta}-e^2_{j\delta}+e^1_{-j\delta}-e^2_{-j\delta})$ and $\mathrm{i}(e^1_{j\delta}-e^2_{j\delta}-e^1_{-j\delta}+e^2_{-j\delta})$ for $j = 1, 3, \ldots$;
4. $(e_{j\delta+\alpha_1}-e_{j\delta+\alpha_2}+e_{-j\delta-\alpha_1}-e_{-j\delta-\alpha_2}-e_{j\delta-\alpha_1}+e_{j\delta-\alpha_2}-e_{-j\delta+\alpha_1}+e_{-j\delta+\alpha_2})$ and $\mathrm{i}(e_{j\delta+\alpha_1}-e_{j\delta+\alpha_2}-e_{-j\delta-\alpha_1}+e_{-j\delta-\alpha_2}-e_{j\delta-\alpha_1}+e_{j\delta-\alpha_2}+e_{-j\delta+\alpha_1}-e_{-j\delta+\alpha_2})$ for $j = 1, 3, \ldots$;

5. $(e_{j\delta+\alpha_1+\alpha_2} + e_{-j\delta-\alpha_1-\alpha_2} - e_{j\delta-\alpha_1-\alpha_2} - e_{-j\delta+\alpha_1+\alpha_2})$ for $j = 1, 3, \ldots$;
6. $\mathrm{i}(e_{j\delta+\alpha_1+\alpha_2} - e_{-j\delta-\alpha_1-\alpha_2} - e_{j\delta-\alpha_1-\alpha_2} + e_{-j\delta+\alpha_1+\alpha_2})$ for $j = 1, 3, \ldots$;

(b) with eigenvalue -1:

1. $(h_{\alpha_1} + h_{\alpha_2})$, $(e_{\alpha_1} + e_{\alpha_2} - e_{-\alpha_1} - e_{-\alpha_2})$;
2. $\mathrm{i}(e^1_{j\delta} + e^2_{j\delta} + e^1_{-j\delta} + e^2_{-j\delta})$ and $(e^1_{j\delta} + e^2_{j\delta} - e^1_{-j\delta} - e^2_{-j\delta})$ for $j = 2, 4, \ldots$;
3. $\mathrm{i}(e_{j\delta+\alpha_1} + e_{j\delta+\alpha_2} + e_{-j\delta-\alpha_1} + e_{-j\delta-\alpha_2} - e_{j\delta-\alpha_1} - e_{j\delta-\alpha_2} - e_{-j\delta+\alpha_1} - e_{-j\delta+\alpha_2})$ and $(e_{j\delta+\alpha_1} + e_{j\delta+\alpha_2} - e_{-j\delta-\alpha_1} - e_{-j\delta-\alpha_2} - e_{j\delta-\alpha_1} - e_{j\delta-\alpha_2} + e_{-j\delta+\alpha_1} + e_{-j\delta+\alpha_2})$ for $j = 2, 4, \ldots$;
4. $\mathrm{i}(e_{j\delta+\alpha_1} - e_{j\delta+\alpha_2} + e_{-j\delta-\alpha_1} - e_{-j\delta-\alpha_2} + e_{j\delta-\alpha_1} - e_{j\delta-\alpha_2} + e_{-j\delta+\alpha_1} - e_{-j\delta+\alpha_2})$ and $(e_{j\delta+\alpha_1} - e_{j\delta+\alpha_2} - e_{-j\delta-\alpha_1} + e_{-j\delta-\alpha_2} + e_{j\delta-\alpha_1} - e_{j\delta-\alpha_2} - e_{-j\delta+\alpha_1} + e_{-j\delta+\alpha_2})$ for $j = 1, 3, \ldots$;
5. $\mathrm{i}(e_{j\delta+\alpha_1+\alpha_2} + e_{-j\delta-\alpha_1-\alpha_2} + e_{j\delta-\alpha_1-\alpha_2} + e_{-j\delta+\alpha_1+\alpha_2})$ for $j = 1, 3, \ldots$;
6. $(e_{j\delta+\alpha_1+\alpha_2} - e_{-j\delta-\alpha_1-\alpha_2} + e_{j\delta-\alpha_1-\alpha_2} - e_{-j\delta+\alpha_1+\alpha_2})$ for $j = 1, 3, \ldots$.

2. *Type 1a involutive automorphisms with $u = -1$ and associated real forms of $A_2^{(2)}$*

There exists a class of involutive automorphisms with representative given by $\mathbf{U}(t) = \mathbf{1}_3$ and $\xi = 0$. For the associated real form the eigenvectors of the corresponding involutive automorphism ψ are:

(i) with eigenvalue $+1$:

(a) $\mathrm{i}c$, $\mathrm{i}d$, $\mathrm{i}(h_{\alpha_1} + h_{\alpha_2})$, $(e_{\alpha_1} + e_{\alpha_2} + e_{-\alpha_1} + e_{-\alpha_2})$, $\mathrm{i}(e_{\alpha_1} + e_{\alpha_2} - e_{-\alpha_1} - e_{-\alpha_2})$;
(b) $(e^1_{j\delta} + e^2_{j\delta} + e^1_{-j\delta} + e^2_{-j\delta})$ and $\mathrm{i}(e^1_{j\delta} + e^2_{j\delta} - e^1_{-j\delta} - e^2_{-j\delta})$ for $j = 2, 4, \ldots$;
(c) $(e_{j\delta+\alpha_1} + e_{j\delta+\alpha_2} + e_{-j\delta-\alpha_1} + e_{-j\delta-\alpha_2})$ and $\mathrm{i}(e_{j\delta+\alpha_1} + e_{j\delta+\alpha_2} - e_{-j\delta-\alpha_1} - e_{-j\delta-\alpha_2})$ for $j = \pm 2, \pm 4, \ldots$;

(ii) with eigenvalue -1:

(a) $\mathrm{i}(e^1_{j\delta} - e^2_{j\delta} + e^1_{-j\delta} - e^2_{-j\delta})$ and $(e^1_{j\delta} - e^2_{j\delta} - e^1_{-j\delta} + e^2_{-j\delta})$ for $j = 1, 3, \ldots$;
(b) $\mathrm{i}(e_{j\delta+\alpha_1} - e_{j\delta+\alpha_2} + e_{-j\delta-\alpha_1} - e_{-j\delta-\alpha_2})$ and $(e_{j\delta+\alpha_1} - e_{j\delta+\alpha_2} - e_{-j\delta-\alpha_1} + e_{-j\delta-\alpha_2})$ for $j = \pm 1, \pm 3, \ldots$;
(c) $\mathrm{i}(e_{j\delta+\alpha_1+\alpha_2} + e_{-j\delta-\alpha_1-\alpha_2})$ and $(e_{j\delta+\alpha_1+\alpha_2} - e_{-j\delta-\alpha_1-\alpha_2})$ for $j = \pm 1, \pm 3, \ldots$.

3. *Type 2a involutive automorphisms with $u = 1$ and associated real forms of $A_2^{(2)}$*

(i) There exists a class of involutive automorphisms with representative given by $\mathbf{U}(t) = \mathbf{1}_3$ and $\xi = 0$. For the associated real form the eigenvectors of the corresponding involutive automorphism ψ are:

(a) with eigenvalue $+1$:

1. $\mathrm{i}(h_{\alpha_1} + h_{\alpha_2})$, $(e_{\alpha_1} + e_{\alpha_2} + e_{-\alpha_1} + e_{-\alpha_2})$, $\mathrm{i}(e_{\alpha_1} + e_{\alpha_2} - e_{-\alpha_1} - e_{-\alpha_2})$;
2. $(e^1_{j\delta} + e^2_{j\delta} + e^1_{-j\delta} + e^2_{-j\delta})$ for $j = 2, 4, \ldots$;
3. $(e^1_{j\delta} - e^2_{j\delta} + e^1_{-j\delta} - e^2_{-j\delta})$ for $j = 1, 3, \ldots$;
4. $(e_{j\delta+\alpha_1} + e_{j\delta+\alpha_2} + e_{-j\delta-\alpha_1} + e_{-j\delta-\alpha_2} + e_{j\delta-\alpha_1} + e_{j\delta-\alpha_2} + e_{-j\delta+\alpha_1} + e_{-j\delta+\alpha_2})$ and $\mathrm{i}(e_{j\delta+\alpha_1} + e_{j\delta+\alpha_2} - e_{-j\delta-\alpha_1} - e_{-j\delta-\alpha_2} - e_{j\delta-\alpha_1} - e_{j\delta-\alpha_2} + e_{-j\delta+\alpha_1} + e_{-j\delta+\alpha_2})$ for $j = 2, 4, \ldots$;
5. $(e_{j\delta+\alpha_1} - e_{j\delta+\alpha_2} + e_{-j\delta-\alpha_1} - e_{-j\delta-\alpha_2} + e_{j\delta-\alpha_1} - e_{j\delta-\alpha_2} + e_{-j\delta+\alpha_1} - e_{-j\delta+\alpha_2})$ and $\mathrm{i}(e_{j\delta+\alpha_1} - e_{j\delta+\alpha_2} - e_{-j\delta-\alpha_1} + e_{-j\delta-\alpha_2} - e_{j\delta-\alpha_1} + e_{j\delta-\alpha_2} + e_{-j\delta+\alpha_1} - e_{-j\delta+\alpha_2})$ for $j = 1, 3, \ldots$;
6. $(e_{j\delta+\alpha_1+\alpha_2} + e_{-j\delta-\alpha_1-\alpha_2} + e_{j\delta-\alpha_1-\alpha_2} + e_{-j\delta+\alpha_1+\alpha_2})$ for $j = 1, 3, \ldots$;
7. $\mathrm{i}(e_{j\delta+\alpha_1+\alpha_2} - e_{-j\delta-\alpha_1-\alpha_2} - e_{j\delta-\alpha_1-\alpha_2} + e_{-j\delta+\alpha_1+\alpha_2})$ for $j = 1, 3, \ldots$;

(b) with eigenvalue -1:

1. c, d;
2. $(e^1_{j\delta} + e^2_{j\delta} - e^1_{-j\delta} - e^2_{-j\delta})$ for $j = 2, 4, \ldots$;
3. $(e^1_{j\delta} - e^2_{j\delta} - e^1_{-j\delta} + e^2_{-j\delta})$ for $j = 1, 3, \ldots$;
4. $\mathrm{i}(e_{j\delta+\alpha_1} + e_{j\delta+\alpha_2} + e_{-j\delta-\alpha_1} + e_{-j\delta-\alpha_2} - e_{j\delta-\alpha_1} - e_{j\delta-\alpha_2} - e_{-j\delta+\alpha_1} - e_{-j\delta+\alpha_2})$ and $(e_{j\delta+\alpha_1} + e_{j\delta+\alpha_2} - e_{-j\delta-\alpha_1} - e_{-j\delta-\alpha_2} + e_{j\delta-\alpha_1} + e_{j\delta-\alpha_2} - e_{-j\delta+\alpha_1} - e_{-j\delta+\alpha_2})$ for $j = 2, 4, \ldots$;
5. $\mathrm{i}(e_{j\delta+\alpha_1} - e_{j\delta+\alpha_2} + e_{-j\delta-\alpha_1} - e_{-j\delta-\alpha_2} - e_{j\delta-\alpha_1} + e_{j\delta-\alpha_2} - e_{-j\delta+\alpha_1} + e_{-j\delta+\alpha_2})$ and $(e_{j\delta+\alpha_1} - e_{j\delta+\alpha_2} - e_{-j\delta-\alpha_1} + e_{-j\delta-\alpha_2} + e_{j\delta-\alpha_1} - e_{j\delta-\alpha_2} - e_{-j\delta+\alpha_1} + e_{-j\delta+\alpha_2})$ for $j = 1, 3, \ldots$;
6. $\mathrm{i}(e_{j\delta+\alpha_1+\alpha_2} + e_{-j\delta-\alpha_1-\alpha_2} - e_{j\delta-\alpha_1-\alpha_2} - e_{-j\delta+\alpha_1+\alpha_2})$ for $j = 1, 3, \ldots$;
7. $(e_{j\delta+\alpha_1+\alpha_2} - e_{-j\delta-\alpha_1-\alpha_2} + e_{j\delta-\alpha_1-\alpha_2} - e_{-j\delta+\alpha_1+\alpha_2})$ for $j = 1, 3, \ldots$.

(ii) There exists a class of involutive automorphisms with representative given by $\mathbf{U}(t) = \mathbf{d}$ and $\xi = 0$, where $\mathbf{d}$ is given by (44). For the associated real form the eigenvectors of the corresponding involutive automorphism ψ are:

(a) with eigenvalue $+1$:

1. $(e_{\alpha_1} + e_{\alpha_2} + e_{-\alpha_1} + e_{-\alpha_2})$;
2. $\mathrm{i}(e^1_{j\delta} + e^2_{j\delta} - e^1_{-j\delta} - e^2_{-j\delta})$ for $j = 2, 4, \ldots$;
3. $(e_{j\delta+\alpha_1} + e_{j\delta+\alpha_2} + e_{-j\delta-\alpha_1} + e_{-j\delta-\alpha_2} + e_{j\delta-\alpha_1} + e_{j\delta-\alpha_2} + e_{-j\delta+\alpha_1} + e_{-j\delta+\alpha_2})$ and $(e_{j\delta+\alpha_1} + e_{j\delta+\alpha_2} + e_{-j\delta-\alpha_1} + e_{-j\delta-\alpha_2} - e_{j\delta-\alpha_1} - e_{j\delta-\alpha_2} - e_{-j\delta+\alpha_1} - e_{-j\delta+\alpha_2})$ for $j = 2, 4, \ldots$;
4. $(e^1_{j\delta} - e^2_{j\delta} + e^1_{-j\delta} - e^2_{-j\delta})$ for $j = 1, 3, \ldots$;
5. $\mathrm{i}(e_{j\delta+\alpha_1} - e_{j\delta+\alpha_2} - e_{-j\delta-\alpha_1} + e_{-j\delta-\alpha_2} + e_{j\delta-\alpha_1} - e_{j\delta-\alpha_2} - e_{-j\delta+\alpha_1} + e_{-j\delta+\alpha_2})$ and $\mathrm{i}(e_{j\delta+\alpha_1} - e_{j\delta+\alpha_2} - e_{-j\delta-\alpha_1} + e_{-j\delta-\alpha_2} - e_{j\delta-\alpha_1} + e_{j\delta-\alpha_2} + e_{-j\delta+\alpha_1} - e_{-j\delta+\alpha_2})$ for $j = 1, 3, \ldots$;
6. $\mathrm{i}(e_{j\delta+\alpha_1+\alpha_2} - e_{-j\delta-\alpha_1-\alpha_2} + e_{j\delta-\alpha_1-\alpha_2} - e_{-j\delta+\alpha_1+\alpha_2})$ for $j = 1, 3, \ldots$;
7. $\mathrm{i}(e_{j\delta+\alpha_1+\alpha_2} - e_{-j\delta-\alpha_1-\alpha_2} - e_{j\delta-\alpha_1-\alpha_2} + e_{-j\delta+\alpha_1+\alpha_2})$ for $j = 1, 3, \ldots$;

(b) with eigenvalue -1:

1. c, d, $(h_{\alpha_1} + h_{\alpha_2})$, $(e_{\alpha_1} + e_{\alpha_2} - e_{-\alpha_1} - e_{-\alpha_2})$;
2. $\mathrm{i}(e^1_{j\delta} + e^2_{j\delta} + e^1_{-j\delta} + e^2_{-j\delta})$ for $j = 2, 4, \ldots$;
3. $(e_{j\delta+\alpha_1} + e_{j\delta+\alpha_2} - e_{-j\delta-\alpha_1} - e_{-j\delta-\alpha_2} + e_{j\delta-\alpha_1} + e_{j\delta-\alpha_2} - e_{-j\delta+\alpha_1} - e_{-j\delta+\alpha_2})$ and $(e_{j\delta+\alpha_1} + e_{j\delta+\alpha_2} - e_{-j\delta-\alpha_1} - e_{-j\delta-\alpha_2} - e_{j\delta-\alpha_1} - e_{j\delta-\alpha_2} + e_{-j\delta+\alpha_1} + e_{-j\delta+\alpha_2})$ for $j = 2, 4, \ldots$;
4. $(e^1_{j\delta} - e^2_{j\delta} - e^1_{-j\delta} + e^2_{-j\delta})$ for $j = 1, 3, \ldots$;
5. $\mathrm{i}(e_{j\delta+\alpha_1} - e_{j\delta+\alpha_2} + e_{-j\delta-\alpha_1} - e_{-j\delta-\alpha_2} + e_{j\delta-\alpha_1} - e_{j\delta-\alpha_2} + e_{-j\delta+\alpha_1} - e_{-j\delta+\alpha_2})$ and $\mathrm{i}(e_{j\delta+\alpha_1} - e_{j\delta+\alpha_2} + e_{-j\delta-\alpha_1} - e_{-j\delta-\alpha_2} - e_{j\delta-\alpha_1} + e_{j\delta-\alpha_2} - e_{-j\delta+\alpha_1} + e_{-j\delta+\alpha_2})$ for $j = 1, 3, \ldots$;
6. $\mathrm{i}(e_{j\delta+\alpha_1+\alpha_2} + e_{-j\delta-\alpha_1-\alpha_2} + e_{j\delta-\alpha_1-\alpha_2} + e_{-j\delta+\alpha_1+\alpha_2})$ for $j = 1, 3, \ldots$;
7. $\mathrm{i}(e_{j\delta+\alpha_1+\alpha_2} + e_{-j\delta-\alpha_1-\alpha_2} - e_{j\delta-\alpha_1-\alpha_2} - e_{-j\delta+\alpha_1+\alpha_2})$ for $j = 1, 3, \ldots$.

References

[1] Cornwell J F 1992 *J. Phys. A: Math. Gen.* **25** 2311–33
[2] Gantmacher F R 1939 *Rec. Math. (Mat. Sbornik) N. S.* **5** (47) 101–44
[3] Gantmacher F R 1939 *Rec. Math. (Mat. Sbornik) N. S.* **5** (47) 217–50
[4] Bernard D 1987 *Nucl. Phys.* B **288** 628–48
[5] Walton M A 1989 *J. Math. Phys.* **30** 1408–14
[6] Bouwknegt P 1989 *J. Math. Phys.* **30** 571–84
[7] Font A 1991 *Mod. Phys. Lett.* A **6** 3265–73
[8] Cornwell J F 1992 *J. Phys. A: Math. Gen.* **25** 2335–58
[9] Cornwell J F 1992 *J. Phys. A: Math. Gen.* **25** 3955–75

[10] Cornwell J F 1992 *J. Phys. A: Math. Gen.* **25** 3977–4015
[11] Clarke S P and Cornwell J F 1994 *J. Phys. A: Math. Gen.* **27** 131–52
[12] Clarke S P and Cornwell J F 1998 to be published
[13] Pati K C and Parashar D 1998 *J. Math. Phys.* **39** 5015–23
[14] Kac V G 1985 *Infinite Dimensional Lie algebras* (Cambridge: Cambridge University Press)
[15] Goddard P and Olive D 1986 *Int. J. Mod. Phys.* A **1** 303–414
[16] Goddard P and Olive D 1988 *Kac–Moody and Virasoro algebras* (Singapore: World Scientific)
[17] Cornwell J F 1989 *Group Theory in Physics* vol III (London: Academic Press)